Some Notes on Numerical Analysis Qualifying Exam

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First some general remarks: Unlike other problems, problem 6 and 7 on the Numerical Analysis qualifying exam test on PDE materials that are mostly not covered in Math-269B. Topics that have been tested on but not covered in 269B include

(1) conservation laws (F22.07, S22.07, S21.06, S20.06, F19.06)

(2) second order hyperbolic equations (F21.07, S20.07)

(3) diffusion equation in 2D/3D or convection-diffusion equation (S22.06, F21.06, F20.07, F19.07, S19.07)

(4) system of transport equations (F20.06, S19.06)

(5) who knows :(

My 269B used the textbook [Str04], which contains useful stuff for (2)-(4). More specifically, Chapter 6 is useful for convection-diffusion equations, Chapter 7 is useful for systems and Chapter 8 is useful for second order hyperbolic equations (although you can usually transform these second order equations into first order equivalent transport systems by considering $\mathbf{v} = \begin{pmatrix} u_t \\ u_x \end{pmatrix}$ which gives you something of the form $\mathbf{v}_t = A\mathbf{v}_x + B\mathbf{v}$). For many (2)-(4) type problems, the first part asks about well-posedness of the PDE, which can be done using symbol analysis (covered in Chapter 9) or Fourier transform.

Conservation law problems appear frequently and are usually very consistent. If you prepare and just remember one example, it can be basically free points. However, the book [LeV08] I used doesn't seem to contain a good and complete solution template, so we (credit to Evan Davis and Jacob Murri) put together a short solution by modifying the one given in Howard Heaton's notes. I typed up this document after taking the NA exam, hoping future students can also benefit.

The conservation law problem almost always takes the following form:

Consider the equation

$$u_t + (f(u))_x = \varepsilon u_{xx}$$

to be solved for $t > 0$, $0 \leqslant x \leqslant 1$ with 1 periodic boundary conditions and smooth initial data $u(x, 0) = u_0(x)$. Give a finite difference scheme that converges for all $t > 0$, even as $\varepsilon \to 0^+$.

Answer: Since Lax-equivalence theorem no longer applies, we need some other theorem. We use Theorem 15.2 in [LeV08]: Suppose there is a numerical scheme that can be written in conservation form with a Lipschitz continuous numerical flux, consistent with some scalar conservation law. If the method is TV-stable, then the method is convergent for all $t > 0$.

We consider the modified Lax-Friedrichs scheme and verify assumptions of the theorem. The scheme is given by

$$\frac{v_m^{n+1} - kv_m^n - \frac{1-k}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + \frac{f(v_{m+1}^n) - f(v_{m-1}^n)}{2h} = \varepsilon \frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2},$$

which can be equivalently written as

$$v_m^{n+1} = \mathcal{H}(v_{m-1}^m, v_m^n, v_{m+1}^n) := kv_m^n + \frac{1-k}{2}(v_{m+1}^n + v_{m-1}^n) - \frac{k}{2h}(f_{m+1}^n - f_{m-1}^n) + \frac{\varepsilon k}{h^2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n).$$

We now verify the assumptions of the theorem.

- The scheme can be written in conservation form, i.e. we can find numerical flux function $F(u, v)$ with
$$v_m^{n+1} = u_m^n - \lambda(F(v_m^n, v_{m+1}^n) - F(v_{m-1}^n, v_m^n))$$
where $\lambda = \frac{k}{h}$. We note
$$\begin{aligned} v_m^{n+1} = v_m^n + \frac{1-k}{2}(v_{m+1}^n - v_m^n - v_m^n + v_{m+1}^n) - \frac{k}{2h}(f_{m+1}^n + f_m^n - f_m^n - f_{m-1}^n) \\ + \frac{\varepsilon k}{h^2}(v_{m+1}^n - v_m^n - v_m^n + v_{m-1}^n). \end{aligned}$$
Thus we can set
$$F(u, v) = \left(\frac{1-k}{2\lambda} + \frac{\varepsilon}{h}\right)(u - v) + \frac{f(u) + f(v)}{2}.$$
- The numerical flux is Lipschitz continuous and consistent with f, i.e., there exists $L > 0$ with
$$|F(u, v) - f(\overline{u})| \leqslant L \cdot \max\{|u - \overline{u}|, |v - \overline{u}|\}$$
for u, v sufficiently close to $\overline{u}$. Assume v is bounded, then f is a C^1 function on a compact set and thus Lipschitz continuous. Assume f is K-Lipschitz, λ is kept fixed and $\varepsilon = \frac{h^2}{2}$ with $h \leqslant 1$, we have
$$\begin{aligned} |F(u, v) - f(\overline{u})| &\leqslant \left(\frac{1}{2\lambda} + \frac{1}{2}\right) \cdot |u - \overline{u} + \overline{u} - v| + \frac{1}{2}|f(u) - f(\overline{u})| + \frac{1}{2}|f(v) - f(\overline{u})| \\ &\leqslant \left(\frac{1}{\lambda} + 1 + K\right) \cdot \max\{|u - \overline{u}|, |v - \overline{u}|\}. \end{aligned}$$

- We finally check the scheme is TV-stable. Howard Heaton's solution checks that the scheme is ℓ_1-contracting, which is a bit long and hard to memorize. Luckily, there is a simpler condition, monotonicity, which is sufficient due to the following implications:

$$\text{monotone} \;\Rightarrow\; \ell_1\text{-contracting} \;\Rightarrow\; \text{TVD (total variation diminishing)}.$$

  Note our scheme is given by

$$v_m^{n+1} = \mathcal{H}(v_{m-1}^m, v_m^n, v_{m+1}^n).$$

  It is monotone if

$$\frac{\partial \mathcal{H}}{\partial v_{m-1}^n} \geqslant 0, \frac{\partial \mathcal{H}}{\partial v_m^n} \geqslant 0, \frac{\partial \mathcal{H}}{\partial v_{m+1}^n} \geqslant 0.$$

  We claim this is true if the CFL condition is satisfied:

$$|\lambda f'(u)| \leqslant 1 \text{ for any } u \text{ satisfying } \min_m(u_m^n, v_m^n) \leqslant u \leqslant \max_m(u_m^n, v_m^n).$$

  We simply compute (remember $\varepsilon = \frac{h^2}{2}$)

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial v_m^n} &= k - \frac{2\varepsilon k}{h^2} = k - k = 0 \\
\frac{\partial \mathcal{H}}{\partial v_{m-1}^n} &= \frac{1-k}{2} + \frac{\lambda}{2} f'(v_{m-1}^n) + \frac{\varepsilon k}{h^2} = \frac{1}{2} - \frac{k}{2} + \frac{\lambda f'}{2} + \frac{k}{2} = \frac{1 + \lambda f'}{2} \geqslant 0 \\
\frac{\partial \mathcal{H}}{\partial v_{m+1}^n} &= \frac{1-k}{2} - \frac{\lambda}{2} f'(v_{m+1}^n) + \frac{\varepsilon k}{h^2} = \frac{1}{2} - \frac{k}{2} - \frac{\lambda f'}{2} + \frac{k}{2} = \frac{1 - \lambda f'}{2} \geqslant 0.
\end{aligned}$$

Now just apply the theorem and note $\varepsilon = \frac{h^2}{2} \to 0$ as $k, h \to 0$.

Remark 0.1. *Occasionally the problem will have a first part asking you to provide a second order accurate scheme which converges for small t. Just give any convergent second order method (e.g. Crank-Nicolson). Part of the reason it doesn't work for all $t > 0$ is Lax-equivalence theorem no longer applies (shocks/rarefactions make PDE no longer well-posed). Note by Theorem 15.6 in [LeV08], our monotone scheme cannot be second order accurate.*

Remark 0.2. *There are times when the problem also asks you to prove minimum/maximum principle of your scheme, which follows directly from monotonicity. Use mean value theorem to get $f(v_{m+1}^n) - f(v_{m-1}^n) = f'(\xi)(v_{m+1}^n - v_{m-1}^n)$ and do the usual L^∞ norm bound.*

Remark 0.3. *Sometimes the PDE doesn't have the artificial viscosity term ($\varepsilon = 0$). In that case the regular Lax-Friedrichs scheme will directly work:*

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + \frac{f(v_{m+1}^n) - f(v_{m-1}^n)}{2h} = 0$$

The modification term kv_m^n when $\varepsilon \neq 0$ is just to make the scheme monotone ($\frac{\partial \mathcal{H}}{\partial v_m^n} \geqslant 0$). The rest of the proof should be very similar.

References

[LeV08] Randall J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhauser Verlag, 2008.

[Str04] John C. Strikwerda. *Finite difference schemes and partial differential equations*. Siam, 2004.